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SINGULAR MANIFOLDS

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1. Introduction

If ϕ: $X \to Y$ is a map of topological spaces and $x \in X$, then ϕ_x will denote the germ of ϕ at x. Let $\mathfrak{F}(p, q) = \{\phi: \boldsymbol{R}^p \to \boldsymbol{R}^q \mid \phi \text{ is } \mathscr{C}^\infty \text{ and } \phi(0) = 0\}$ and let $J(p, q) = \{\phi_0 \mid \phi \in \mathfrak{F}(p, q)\}$. If $\phi \in \mathfrak{F}(p, q)$ or $\phi \in J(p, q)$, then $[\phi]^n$ will denote the set of germs at the origin of elements of $\mathfrak{F}(p, q)$, which agree with ϕ up to and including order n at the origin. $[\phi]^n$ will occasionally be abbreviated to ϕ. Let $J^n(p, q) = \{[\phi]^n \mid \phi \in J(p, q)\}$.

Whenever m is an integer, $\mathscr{L}_m$ will denote the set of invertible germs in $J(m, m)$. $\mathscr{L}_m$ is a group. Furthermore, there is a group action of $\mathscr{L}_p \times \mathscr{L}_q$ on $J^n(p, q): (\alpha, \beta)([\phi]^n) = [\beta\phi\alpha^{-1}]^n$. Suppose ϕ: $U \to \boldsymbol{R}^q$ is $\mathscr{C}^\infty$ where U is an open subset of $\boldsymbol{R}^p$. Define t_ϕ: $U \to J(p, q)$ by $t_\phi(x)$ is the germ at the origin of $y \to \phi(x + y) - \phi(x)$. In the following all manifolds are $\mathscr{C}^\infty$ and paracompact, and all maps are $\mathscr{C}^\infty$.

Let $\tilde{\mathscr{L}}_m$ be a subgroup of $\mathscr{L}_m$. Suppose M is an m-dimensional manifold and $\mathscr{A}$ is an atlas of coordinate functions for M. The pair $(M, \mathscr{A})$ will be called a manifold of type $\tilde{\mathscr{L}}_m$ if for all $x \in M$ and coordinate functions $\alpha_1, \alpha_2 \in \mathscr{A}$ whose domains contain x, $t_{\alpha_2\alpha_1^{-1}}(\alpha_1(x)) \in \tilde{\mathscr{L}}_m$. The atlas $\mathscr{A}$ will be suppressed from the notation.

Let X be a p-manifold and Y a q-manifold. $J^n(X, Y)$ will be the bundle with base $X \times Y$, fiber $J^n(p, q)$, and group $\mathscr{L}_p \times \mathscr{L}_q$. Let $\tilde{\mathscr{L}}_p$ be a subgroup of $\mathscr{L}_p$ and $\tilde{\mathscr{L}}_q$ a subgroup of $\mathscr{L}_q$. Suppose X is a manifold of type $\tilde{\mathscr{L}}_p$ and Y is a manifold of type $\tilde{\mathscr{L}}_q$. Then the group of $J^n(X, Y)$ is reducible to $\tilde{\mathscr{L}}_p \times \tilde{\mathscr{L}}_q$. $J^n(X, Y)$ may be looked at as the set of equivalence classes of germs of maps of X into Y where two germs are equivalent if they agree up to order n.

If f: $X \to Y$ and $x \in X$, then $f^n(x)$ will denote the equivalence class containing the germ of f at x. Thus a map f: $X \to Y$ induces a commutative triangle:

$$\begin{array}{ccc} & & J^n(X, Y) \\ & \overset{f^n}{\nearrow} & \downarrow \\ X & \xrightarrow[(id, f)]{} & X \times Y \end{array}$$

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Let $A \subset J^n(p, q)$ and let A be invariant under $\tilde{\mathscr{L}}_p \times \tilde{\mathscr{L}}_q$. Then $J^n_A(X, Y)$ will denote the bundle with base $X \times Y$, fiber A, and group $\tilde{\mathscr{L}}_p \times \tilde{\mathscr{L}}_q$. Suppose A is as above and $f: X \to Y$. Define $A(f)$, the singular set of f of type A, to be the set $(f^n)^{-1} J^n_A(X, Y)$. If A is a manifold, then so is $J^n_A(X, Y)$. If A is a manifold and f is such that f^n is transversal to $J^n_A(X, Y)$, then f will be called A-transversal. If f is A-transversal, then $A(f)$ is a submanifold of X and, furthermore, the codimension of $A(f)$ in X is the codimension of A in $J^n(p, q)$.

Let $\mathscr{C}^{n+1}(X, Y)$ denote the set of $\mathscr{C}^\infty$ maps of X into Y, provided with the topology of compact convergence of all partials of order less than or equal to $n + 1$.

The Thom transversality theorem states that if B ia a submanifold of $J^n(X, Y)$, then the set of maps $f: X \to Y$ such that f^n is transversal to B is a Baire set in $\mathscr{C}^{n+1}(X, Y)$. If X is compact, then this set is open and dense. (See [3] for a proof of the transversality theorem.) Thus, if $A \subset J^n(p, q)$ is a manifold and is invariant under $\tilde{\mathscr{L}}_p \times \tilde{\mathscr{L}}_q, X$ is a manifold of type $\tilde{\mathscr{L}}_p$ and Y is a manifold of type $\tilde{\mathscr{L}}_q$, then $A(f)$ is a manifold for a large class of functions $f: X \to Y$.

One thing which makes this interesting is that, in general, for A-transversal f there are connections between $A(f)$ and global properties of X and Y. For example, if $A = \{[0]^1\} \subset J^1(p, 1)$, X is a compact p-manifold, $Y = \boldsymbol{R}$ and f is A-transversal, then the Morse theory tells us how to predict global properties of X from the behavior of f in a neighborhood of $A(f)$. Other results in this direction are proven in [2], [4], and [5]. Further (rather incomplete) results will be presented here but the main result of this paper is the construction of submanifolds of $J^n(p, q)$ which are invariant under various subgroups $\tilde{\mathscr{L}}_p \times \tilde{\mathscr{L}}_q$ of $\mathscr{L}_p \times \mathscr{L}_q$.

2. Grassmann bundles

If E is a bundle over X and $x \in X$, then E_x will denote the fiber of E over x. If $A \subset X$, then the restriction of E to A will also be written E. If F is a bundle over Y and $h: E \to F$, then $h_x: E_x \to F$ will denote the restriction of h to E_x. If $f: X \to Y$ is a map of manifolds, then $Tf: TX \to TY$ will denote the corresponding map of tangent bundles. If A is a submanifold of X, then $T(X, A)$ will denote the normal bundle of A in X. Finally, if E is a vector bundle over X, then X will be identified with the image of the zero section of E. Propositions 2.1 and 2.2 are written up similarly in [5].

Proposition 2.1. *Let $f: X \to Y$ and let N be a submanifold of Y. If f is transversal to N, then Tf induces a map $T(X, f^{-1}N) \to T(Y, N)$ which restricts to isomorphisms of fibers.*

Proof. The desired mapping is given in the following exact commutative diagram:

$$\begin{array}{ccccccccc} 0 \to & T(f^{-1}N) & \to & TX & \to & T(X, f^{-1}N) & \to 0 & & \text{over } f^{-1}N \\ & \downarrow & & \downarrow & & \downarrow & & & \\ 0 \to & TN & \to & TY & \to & T(Y, N) & \to 0 & & \text{over } N \end{array}$$

That the mapping induces epimorphisms of fibers is a restatement of the transversality of f, and that it is $1:1$ on fibers follows from dimensional considerations. q.e.d.

Suppose E is a vector bundle over X and $\sigma: X \to E$ is a section. Then σ will be called a transversal section of E if it is transversal to X (the image of the zero section of E).

Let E be a vector bundle over X. Then $T(E, X)$ is equivalent to E over X. Thus, if $\sigma: X \to E$ is a transversal section of E, then $T(X, \sigma^{-1}X)$ is equivalent to E over X.

Let E be an m-dimensional vector bundle over X and let $a \leq m$. Define $G_a(E) = \{\underline{p} \mid \underline{p}$ is an a-dimensional subspace of some fiber of $E\}$. Structure for $G_a(E)$ as a bundle over X is induced by that of E. Let $\bar{\pi}: G_a(E) \to X$ be the bundle projection.

Define a vector bundle L_a over $G_a(E)$ by $L_a = \{(\underline{p}, v) \mid v \in \underline{p}\}$. Define M_a, an $(m -$ a)-dimensional bundle over $G_a(E)$, by the exactness of $0 \to L_a \to \bar{\pi}^*E \to M_a \to 0$.

Proposition 2.2. *Let Z be a submanifold of X and let $s: Z \to G_a(E)$ be a section. Then over sZ, $T(\bar{\pi}^{-1}Z, sZ) \approx L_a^* \otimes M_a$, where L_a^* denotes the dual of L_a.*

Proof. Define a vector bundle F over $\bar{\pi}^{-1}Z$ (and a morphism ψ) by the exactness of $0 \to \bar{\pi}^*s^*L_a \to \bar{\pi}^*E \xrightarrow{\psi} F \to 0$. Over $\bar{\pi}^{-1}Z$ there is a bunble morphism $L_a \to F$ given by the composition $L_a \to \bar{\pi}^*E \to F$. This morphism induces a section η of $L_a^* \otimes F$ over $\bar{\pi}^{-1}Z$. Furthermore, sZ is the zero set of η. If η is a transversal section of $L_a^* \otimes F$ then, by Proposition 2.1, $T(\bar{\pi}^{-1}Z, sZ) \approx L_a^* \otimes F$ over sZ. Since $F = M_a$ over sZ, it suffices to demonstrate the transversality of η.

Let $x \in Z$ and let $\alpha_1, \cdots, \alpha_m$ be a vector space basis for E_x such that $s(x)$ is the span of $\alpha_1, \cdots, \alpha_a$. Any a-plane $\underline{p}$ in $G_a(E)_x$ near $s(x)$ is uniquely expressible as the span of a vectors, $\alpha_1 + v_{1,1}(\underline{p})\alpha_{a+1} + \cdots + v_{1,m-a}(\underline{p})\alpha_m, \cdots, \alpha_a + v_{a,1}(\underline{p})\alpha_{a+1} + \cdots + v_{a,m-a}(\underline{p})\alpha_m$. Thus coordinates $\{v_{i,j}\}$ for $G_a(E)_x$ at $s(x)$ have been fixed.

$$T\eta_{s(x)}\left(\frac{\partial}{\partial v_{i,j}}\right) = \frac{\partial}{\partial v_{i,j}} + ((\mathrm{id} \otimes \psi)(s(x), \alpha_i^* \otimes \alpha_{a+j}))_{s(x)} .$$

Since $\{(\mathrm{id} \otimes \psi)(s(x), \alpha_i^* \otimes \alpha_{a+j}) \mid 1 \leq i \leq a \text{ and } 1 \leq j \leq m - a\}$ is a basis for $(L_a^* \otimes F)_{s(x)}$, the result follows.

3. Fixing the rank of vector bundle morphisms

Let A be a manifold, E_1 a bundle over A, E_2 and E_3 be vector bundles over A, and $\gamma\colon E_1 \to E_2^* \otimes E_3$ a morphism of fiber bundles over A, which induces the identity on A. Suppose $\pi\colon E_1 \to A$ is the bundle projection. Whenever $e \in E_1, \gamma(e) \in (E_2^* \otimes E_3)_{\pi(e)}$ and therefore there is a linear map $(E_2)_{\pi(e)} \to (E_2)_{\pi(e)}$ which corresponds to $\gamma(e)$. Suppose a is not greater than the ffber dimension of E_2 and let $A_a(\gamma) = \{e \in E_1 \mid \text{kernel } \gamma(e) \text{ has dimension a}\}$. In this section we will study the set $A_a(\gamma)$.

Let $\bar{\pi}\colon G_a(\pi^*E_2) \to E_1$ be the bundle projection. Over $G_a(\pi^*E_2)$ there is an exact sequence $0 \to L_a \to \bar{\pi}^*\pi^*E_2 \to M_a \to 0$ as in § 2. We define a section $\gamma_a\colon G_a(\pi^*E_2) \to L_a^* \otimes \bar{\pi}^*\pi^*E_3$ as follows: An element of $G_a(\pi^*E_2)$ is a pair $(e, \underline{p})$ where $e \in E_1$ and $\underline{p}$ is an a-dimensional subspace of $(E_2)_{\pi(e)}$. Let $\gamma_a(e, \underline{p}) = (e, \underline{p}, \eta(e, \underline{p}))$ where $\eta(e, \underline{p})$ is the restriction of $\gamma(e)$ to $\underline{p}$. $\gamma_a(e, \underline{p})$ may be viewed as an element of $(L_a^* \otimes \bar{\pi}^*\pi^*E_3)_{(e,\underline{p})}$.

Definition 3.1. Suppose that there are a vector space V and for each $x \in A$ a diffeomorphism $\theta_x\colon V \to (E_1)_x$ such that $\gamma_x \circ \theta_x$ is linear. γ will be called a-uniform if for all choices of $x_i \in A$ and $\underline{p}_i \in G_a(E_2)x_i$, $i \in \{1, 2\}$, dimension $\{\eta(e, \underline{p}_1) \mid e \in (E_1)_{x_1}\} =$ dimension $\{\eta(e, \underline{p}_2 \mid e \in (E^1)_{x_2}\}$.

$\gamma\colon E_1 \to E_2^* \otimes E_3$ induces $\gamma^a\colon \bar{\pi}^*\pi^*E_1 \to L_a^* \otimes \bar{\pi}^*\pi^*E_3$ as follows: An element of $\bar{\pi}^*\pi^*E_1$ is a triple $(e, \underline{p}, \bar{e})$ where e and $\bar{e}$ are elements of E_1 with $\pi(e) = \pi(\bar{e})$ and $\underline{p}$ is an a-plane in $(E_2)_{\pi(e)}$. Define γ^a by $\gamma^a(e, \underline{p}, \bar{e}) = (e, \underline{p}, \eta(\bar{e}, p))$.

Let $S_a = \gamma^a(\bar{\pi}^*\pi^*E_1)$, and note that the image of the section γ_a is contained in S_a. If γ is a-uniform, then S_a is a vector sub-bundle of $L_a^* \otimes \bar{\pi}^*\pi^*E_3$.

If V is a vector space, $x, y \in V$, and $g\colon \boldsymbol{R} \to V$ is defined by $g(t) = x + ty$, then we define $y_x \in TV_x$ by $y_x = g'(0)$. $TV = \{y_x \mid x, y \in V\}$.

Let V and θ_x be as in Definition 3.1. Since $\gamma_x \circ \theta_x$ is linear, $T(\gamma_x \circ \theta_x)(y_z) = (\gamma_x \circ \theta_x(y))_{\gamma_x \circ \theta_x(z)}$. Now, if $\underline{p} \in G_a(E_2)_x$, then $(S_a)_{\underline{p}}$ is the set of all restrictions to $\underline{p}$ of maps of the form $\gamma_x \circ \theta_x(y)$ where $y \in V$. It follows that if γ is a-uniform, then γ_a is a transversal section of S_a.

Define a vector bundle K_a over $A_a(\gamma)$ by the exactness of $0 \to K_a \to \pi^*E_1 \xrightarrow{\tilde{\gamma}} \pi^*E_3$ where $\tilde{\gamma}$ is defined in the obvious way. (An element of π^*E_2 is a pair (e_1, e_2) where $e_1 \in E_1$ and $e_2 \in (E_2)_{\pi(e_1)}$. Define $\tilde{\gamma}$ by $\tilde{\gamma}(e_1, e_2) = (e_1, \gamma(e_1)e_2)$, an element of π^*E_3.) Define a bundle N_a over $A_a(\gamma)$ by the exactness of $0 \to K_a \to \pi^*E_2 \to N_a \to 0$. Finally, define a section $s_a\colon A_a(\gamma) \to G_a(\pi^*E_2)$ by $s_a(e) = (e, \text{kernel } \gamma(e))$.

Theorem 3.2. *Let $\gamma\colon E_1 \to E_2^* \otimes E_3$ be a-uniform. Then $A_a(\gamma)$ is a submanifold of E_1, and furthermore over $A_a(\gamma)$ there is an exact sequence*

$$0 \to K_a^* \otimes N_a \to s_a^* S_a \to T(E_1, A_a(\gamma)) \to 0 .$$

Proof. The first statement is straightforward and will be treated first. Let W be the zero set of the section γ_a. Since γ_a is a transversal section of S_a, W is a submanifold of $G_a(\pi^*E_2)$. It is easily seen that $s_a A_a(\gamma)$ is an open subset of W. $((e, \underline{p}) \in W$ if and only if $\underline{p} \subset \ker (\gamma(e)$, Thus $s_a A_a(\gamma) \subset W$. If $e \in A_a(\gamma)$ and if $\tilde{e}$ is sufficiently close to e, then dimension $\ker \gamma(\tilde{e})$ is not larger than a. That $s_a A_a(\gamma)$ is open in W follows.) Thus $s_a A_a(\gamma)$ and therefore $A_a(\gamma)$ is a manifold. We now prove the second statement.

Since γ_a is transversal and $s_a A_a(\gamma)$ is open in W, Proposition 2.1 shows that there is an equivalence $T(G_a(\pi^* E_2), s_a A_a(\gamma)) \to S_a$ over $s_a A_a(\gamma)$ induced by $T\gamma_a$, and also that over $s_a A_a(\gamma)$ we have an exact sequence $0 \to L_a \to \bar{\pi}^*\pi^* E_2 \to \bar{\pi}^*\pi^* E_3$ which determines a monomorphism $M_a \to \bar{\pi}^*\pi^* E_3$ and hence a monomorphism $L_a^* \otimes M_a \to L_a^* \otimes \bar{\pi}^*\pi^* E_3$ over $s_a A_a(\gamma)$.

It is not hard to show that the following diagram is communative:

$$\begin{array}{ccc} T(\bar{\pi}^{-1}A_a(\gamma), s_a A_a(\gamma)) & \longrightarrow & T(G_a(\pi^* E_2), s_a A_a(\gamma)) \\ \wr\wr & & \downarrow \\ L_a^* \otimes M_a & \longrightarrow & L_a^* \otimes \bar{\pi}^*\pi^* E_3 \end{array} \qquad \text{over } s_a A_a(\gamma) .$$

Since the image of $T(G_a(\pi^* E_2), s_a A_a(\gamma)) \to L_a^* \otimes \bar{\pi}^*\pi^* E_3$ is contained in the sub-bundle S_a of $L_a^* \otimes \bar{\pi}^*\pi^* E_3$, the image of $L_a^* \otimes M_a$ is contained in S_a. Thus over $s_a A_a(\gamma)$ we have an exact commutative diagram:

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 \to & & T(\bar{\pi}^{-1}A_a(\gamma), s_a A_a(\gamma)) & \to & T(G_a(\pi^* E_2), s_a A_a(\gamma)) & \to & T(G_a(\pi^* E_2), \bar{\pi}^{-1}A_a(\gamma)) \to 0 \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & L_a^* \otimes M_a & \longrightarrow & S_a & & \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

and hence an exact sequence $0 \to L_a^* \otimes M_a \to S_a \to T(G_a(\pi^* E_2), \bar{\pi}^{-1}A_a(\gamma)) \to 0$. Since $s_a^* L_a = K_a, s_a^* M_a = N_a$ and $s_a^* T(G_a(\pi^* E_2), \bar{\pi}^{-1}A_a(\gamma)) = T(E_1, A_a(\gamma))$, the result follows. q.e.d.

Suppose that X and Y are topological spaces and that a group H acts on both X and Y. Let $f \colon X \to Y$. Then f will be called equivariant if for each $h \in H, hf = fh$.

Definition 3.3. Suppose U is a vector bundle over X and there is a group H which acts on U and X in such a way that the bundle projection of U is equivariant. Suppose also that for each $h \in H$ and $x \in X$, $h_x \colon U_x \to U_{h(x)}$ is a vector space isomorphism. Then U will be called an H-bundle.

Proposition 3.4. *Let U_1 and U_2 be H-bundles over X, and suppose H acts on a space Y and $f \colon Y \to X$ is equivariant.*

a) *Then there is a group action of H on U_1^*, which makes U_1^* an H-bundle*;

b) *similarly with $U_1 \otimes U_2$*;

c) *similarly with $f^* U_1$.*

d) *If $U_1 \subset U_2$ and the inclusion is equivariant, then the factor bundle of U_2 by U_1 is an H-bundle.*

e) *If a is not greater than the fiber dimension of U_1, then there is an action* *of H on $G_a(U_1)$ which makes the projection $\bar{\pi}: G_a(U_1) \to X$ equivariant.*

f) *The action of H on $\bar{\pi}^*U_1$ restricts to an action on L_a, which makes L_a an H-bundle over $G_a(U_1)$.*

g) *If H acts differentiably on X (assumed to be a manifold), then TX may be given the structure of an H-bundle.*

h) *If H acts differentiably on Y and X, then $Tf: TY \to TX$ is equivariant.*

Proof. a) The action of h on U_1^* is the dual of the action of h^{-1} on U_1.

b) The action of h on $U_1 \otimes U_2$ is the tensor product of the actions of h on the U_i.

c) An element of f^*U_1 is a pair (y, u) where $u \in U_{1_{f(y)}}$. Define the action of h by $h(y, u) = (hy, hu)$.

e) Since $h \in H$ restricts to vector space isomorphisms of fibers, it takes a-planes into a-planes.

g) The action of h on TX is the derivative of the action of h on X.

Corollary 3.5. *Let E_2 and E_3 be H-bundles over A, and let H act on E_1 in such a way that $\pi: E_1 \to A$ is equivariant. Suppose $\gamma: E_1 \to E_2^* \otimes E_3$ is a-uniform and equivariant. Then $A_a(\gamma)$ is invariant under H. Furthermore the bundles K_a, N_a, $s_a^*S_a$ and $T(E_1, A_a(\gamma))$ are all H-bundles over $A_a(\gamma)$, and the sequence $0 \to K_a^* \otimes N_a \to s_a^*S_a \to T(E_1, A_a(\gamma)) \to 0$ is an exact sequence of equivariant maps.*

Proof. The equivalences $T(\bar{\pi}^{-1}A_a(\gamma), s_aA_a(\gamma)) \to L_a^* \otimes M_a$ and $T(G_a(\pi^*E_2), s_aA_a(\gamma)) \to S_a$ over $s_aA_a(\gamma)$ are induced by derivatives of equivariant maps. The result is now trivial from Proposition 3.4 and the proof of Theorem 3.2.

4. Invariant submanifolds of $J^{n+1}(p, q)$

Fix subgroups $\tilde{\mathscr{L}}_p \subset \mathscr{L}_p$ and $\tilde{\mathscr{L}}_q \subset \mathscr{L}_q$, and let $H = \tilde{\mathscr{L}}_p \times \tilde{\mathscr{L}}_q$.

Let A be a submanifold of $J^n(p, q)$ and suppose A is invariant under H. Let $E_1 = \{[\phi]^{n+1} \mid [\phi]^n \in A\}$. H acts on E_1 in such a way that the projection $\pi: E_1 \to A$ is equivariant.

If U is an open subset of $R^p, f: U \to R^q$ and $x \in U$, then define a linear map $Df: R^p \to R^q$ by $Tf(v_x) = (Df_x(v))_{f(x)}$. Df will abbreviate Df_0.

H acts on $A \times R^p$; $(\alpha, \beta)([\phi]^n, v) = ([\beta\phi\alpha^{-1}]^n, D\alpha(v))$. Let E_2 be a vector sub-bundle of $A \times R^p$, invariant under H. E_2 is an H-bundle over A.

Note that $J^0(p, q) = \{0\}$. Define $\tilde{J}^0(p, q) = R^q$ and $\tilde{J}^m(p, q) = \{[\phi]^m \mid [\phi]^{m-1} = 0\}$ for $m \geq 1$. Define an action of H on $\tilde{J}^0(p, q)$ by $(\alpha, \beta)(w) = D\beta(w)$ and an action of H on $\tilde{J}^m(p, q), m \geq 1$, by $(\alpha, \beta)([\phi]^m) = [\beta\phi\alpha^{-1}]^m$.

Let B be a vector sub-bundle of $A \times \tilde{J}^n(p, q)$ which is invariant under H. Define E_3 by the exactness of $0 \to B \to A \times \tilde{J}^n(p, q) \to E_3 \to 0$. E_3 is an H-bundle over A.

We now proceed to define a bundle morphism $\gamma: E_1 \to E_2^* \otimes E_3$.

If m is an integer and $1 \leq \nu \leq m$, let $\delta(\nu) = (0, \cdots, 0, 1, 0, \cdots, 0) \in R^m$ where the 1 occurs in the ν^{th} position. Let $\omega = (i_1, \cdots, i_p)$ be a tuple of non-negative integers. Define $|\omega| = i_1 + \cdots + i_p$ and $\omega! = i_1! \cdots i_p!$. If $\phi \in \mathfrak{F}(p, 1)$, let $D_\omega \phi = (\partial^{|\omega|}\phi / \partial x_1^{i_1} \cdots \partial x_p^{i_p})(0)$. If $1 \leq j \leq q$, define $u(\omega, j) \in \mathfrak{F}(p, q)$ by $u(\omega, j)(x_1, \cdots, x_p) = (1/\omega!) x_1^{i_1} \cdots x_p^{i_p} \delta(j)$.

If $n \geq 0$, define $H^{n+1}: E_1 \to A \times (R^{p*} \otimes \tilde{J}^n(p, q))$ by

$$H^{n+1}([\phi]^{n+1}) = \left([\phi]^n, \sum_{|\omega|=n, \nu=1, j=1}^{p,q} D_{\omega+\delta(\nu)} \phi_j \delta(\nu)^* \otimes u(\omega, j)\right),$$

where ϕ_j denotes the j^{th} coordinate function of ϕ.

The injection $E_2 \to A \times R^p$ and the epimorphism $A \times \tilde{J}^n(p, q) \to E_3$ together induce an epimorphism $\varepsilon: A \times (R^{p*} \otimes \tilde{J}^n(p, q)) \to E_2^* \otimes E_3$. Define $\gamma: E_1 \to E_2^* \otimes E_3$ by $\gamma = \varepsilon H^{n+1}$.

Motivational remarks. If $\phi \in \mathfrak{F}(p, q)$, let u_ϕ denote the projection of $\phi^n: R^p \to J^n(R^p, R^q) = R^p \times R^q \times J^n(p, q)$ onto $J^n(p, q)$. $J^n(p, q)$ is a vector space, so if $\psi, \tilde{\phi} \in J^n(p, q)$ then $\tilde{\phi}_\psi \in TJ^n(p, q)$. Motivation for studying the map γ comes from the fact that if $(a_1, \cdots, a_p) \in R^p$ then

$$Tu_\phi(a_1, \cdots, a_p)_0 = \left(\sum_{1 \leq |\omega| \leq n, \nu, j} a_\nu D_{\omega+\delta(\nu)} \phi_j u(\omega, j)\right)_{[\phi]^n}.$$

Thus γ is induced by Tu_ϕ and hence $T\phi^n$ but somewhat artificially. Proper selection of A, E_2 and E_3 makes the correspondence $T\phi^n \to \gamma([\phi]^{n+1})$ "natural". Theorem 4.3 and Proposition 4.4 establish criteria for this to be so. If ϕ is A-transversal then $T\phi^n$ determines $TA(\phi)_0$. Sometimes (see Proposition 4.5) γ will carry enough information to determine whether $([\phi]^n, v) \in E_2$ is such that $v_0 \in TA(\phi)_0$. This is central to much of what follows and is the main idea of the proof of Boardman's result, Theorem 6.2.

If V is a vector space, then $\bigodot_m V$ will denote the m-fold symmetric product of V with itself, and $\bigotimes_m V$ denotes the appropriate tensor product so that $\bigodot_m V \subset \bigotimes_m V$.

If $n \geq 0$, there is a vector space isomorphism $\mu_n: \tilde{J}^n(p, q) \to \left(\bigodot_n R^{p*}\right) \otimes R^q$ determined by the equations

$$\mu_n(u((i_1, \cdots, i_p), j)) = \sum_{x \in I} \delta(k(1))^* \otimes \cdots \otimes \delta(k(n))^* \otimes \delta(j),$$

where $I = \{k: \{1, \cdots, n\} \to \{1, \cdots, p\} \mid k^{-1}\{\lambda\}$ has i_λ elements whenever $1 \leq \lambda \leq p\}$.

The notation in the following is as in § 3.

Let $\varepsilon_a: G_a(\pi^* E_2) \times (R^{p*} \otimes \tilde{J}^n(p, q)) \to L_a^* \otimes \bar{\pi}^* \pi^* E_3$ be the epimorphism. Define $\tilde{S}_a = \varepsilon_a\left(G_a(\pi^* E_2) \times (\mathrm{id} \otimes \mu_n)^{-1}\left(\left(\bigodot_{n+1} R^{p*}\right) \otimes R^q\right)\right)$.

Proposition 4.1. $S_a = \tilde{S}_a$.

Proof. The result follows if $H^{n+1}E_1 = A \times (\mathrm{id} \otimes \mu_n)^{-1}\left(\left(\bigodot_{n+1} R^{p*}\right) \otimes R^q\right)$.

That $H^{n+1}E_1 \subset A \times (\mathrm{id} \otimes \mu_n)^{-1}\left(\left(\bigcirc_{n+1} \boldsymbol{R}^{p*}\right) \otimes \boldsymbol{R}^q\right)$ is apparent from the symmetries of $(n+1)^{st}$ order derivatives. The opposite inclusion is equally simple.
q.e.d.

Thus there is a sense in which S_a is the "symmetric subset" of $L_a^* \otimes \pi^*\pi^*E_3$. The condition that γ is a-uniform is the condition that the symmetric subspace of $(L_a^* \otimes \pi^*\pi^*E_3)_{\underline{p}}$ does not depend on the choice of $\underline{p} \in G_a(\pi^*E_2)$.

If $1 \leq m \leq n$, define $C_m\colon J^n(p, q) \to \tilde{J}^m(p, q)$ by

$$C_m([\phi]^n) = \sum_{|\omega|=m, j} D_\omega \phi_j u(\omega, j) .$$

Recall that if $\phi \in \mathfrak{F}(p, q)$, then $t_\phi\colon \boldsymbol{R}^p \to J(p, q)$ is defined by: $t_\phi(x)$ is the germ at the origin of $\phi(x + \cdot) - \phi(x)$. If $m \geq 1$, then t_ϕ induces $t_{\phi m}\colon \boldsymbol{R}^p \to J^m(p, q)$. Note that $\gamma([\phi]^{n+1})([\phi]^n, v)$, is the projection of $([\phi]^n, C_n D t_{\phi n}(v))$ on E_3.

Definition 4.2. Let $C \subset J(p, q)$(or $C \subset J^m(p, q)$). C will be called translation invariant if, for all $\phi \in \mathfrak{F}(p, q)$, $t_\phi^{-1}(C)$ (or $t_{\phi m}^{-1}(C)$) is an open subset of $\boldsymbol{R}^p$.

Whenever $m \geq 1$, there is a linear map inj $(m) =$ inj: $J^m(p, q) \to J^{m+1}(p, p)$ determined by the equations inj $(u(\omega, j)) = u(\omega, j)$.

Theorem 4.3. *Let $\tilde{\mathscr{L}}_p$, $\tilde{\mathscr{L}}_q$, A, B, E_1, E_2, E_3 and $\gamma\colon E_1 \to E_2^* \otimes E_3$ be as above, and suppose, in addition, that $\tilde{\mathscr{L}}_p$ and $\tilde{\mathscr{L}}_q$ are translation invariant. Then γ is equivariant if the following two conditions are met:*

i) $n = 0, n = 1$ *or* $(\mathrm{inj}\,(Dt_{\phi n-1}(v))_{[\phi]^n} \in TA$, *whenever* $([\phi]^n, v) \in E_2$.

ii) $n = 0$ *or* $([\phi]^n, C_n[\phi]^n) \in B$, *whenever* $([\phi]^n)_{[\phi]^n} \in TA$.

Proof. It suffices to show that whenever $\alpha \in \tilde{\mathscr{L}}_p$ and $\beta \in \tilde{\mathscr{L}}_q$ the following two squares are commutative:

$$\begin{array}{ccc} E_1 & \xrightarrow{\gamma} & E_2^* \otimes E_3 \\ \downarrow{\scriptstyle(\alpha^{-1},\mathrm{id})} & & \downarrow{\scriptstyle(\alpha^{-1},\mathrm{id})} \\ E_1 & \xrightarrow{\gamma} & E_2^* \otimes E_3 \end{array} \qquad \begin{array}{ccc} E_1 & \xrightarrow{\gamma} & E_2^* \otimes E_3 \\ \downarrow{\scriptstyle(\mathrm{id},\beta)} & & \downarrow{\scriptstyle(\mathrm{id},\beta)} \\ E_1 & \xrightarrow{\gamma} & E_2^* \otimes E_3 \end{array}$$

We show that the first of these is commutative, the other demonstration being similar.

The commutativity of the square will follow if we can show that if $[\phi]^{n+1} \in E_1$ and $v = (a_1, \cdots, a_p)$ is such that $([\phi]^n, v) \in E_2$, then

$$(*) \qquad \Big([\phi\alpha]^n, \sum_{|\omega|=n, i, j, k} D_{\omega+\delta(k)}(\phi_j\alpha) D_{\delta(i)}(\alpha^{-1})_k a_i u(\omega, j) - R_\alpha \sum_{|\omega|=n, i, j} D_{\omega+\delta(i)}(\phi_j) a_i u(\omega, j)\Big) \in B ,$$

where R_α denotes right composition with α; left composition will be written in the obvious way.

If $\eta = (i_1, \cdots, i_p)$ and $1 \leq j \leq p$, define $v(\eta, j)\colon \boldsymbol{R}^p \to \boldsymbol{R}^p$ by $v(\eta, j)(x) = \dfrac{1}{n!}x_1^{i_1}\cdots x_p^{i_p}\delta(j)$, so $v(\eta, j) \in J(p, p)$. If $1 \leq j \leq q$ and ω is a p-tuple of integers, define $P(\omega, j)\colon J(p, q) \times J(p, p) \to \boldsymbol{R}$ by $P(\omega, j)(\psi, \rho) = D_\omega(\psi_j \rho)$. $\dfrac{\partial P(\omega, j)}{\partial u(\eta, k)}(\psi, \rho)$ and $\dfrac{\partial P(\omega, j)}{\partial v(\eta, k)}(\psi, \rho)$ denote the appropriate partial derivatives evaluated at (ψ, ρ). It follows from the chain rule that

$$D_{\omega+\delta(k)}(\phi_j\alpha) = \sum_{|\eta|\leq|\omega|,\nu} \frac{\partial P(\omega, j)}{\partial u(\eta, j)}(\phi, \alpha) D_{\eta+\delta(\nu)}\phi_j D_{\delta(k)}\alpha_\nu + \sum_{|\eta|\leq|\omega|,\nu} \frac{\partial P(\omega, j)}{\partial v(\eta, \nu)}(\psi, \alpha) D_{\eta+\delta(k)}\alpha_\nu \,.$$

Thus

$$\begin{aligned}
&\sum_{|\omega|=n,i,j,k} D_{\omega+\delta(k)}(\phi_j\alpha) D_{\delta(i)}(\alpha^{-1})_k a_i u(\omega, j) \\
&= \sum_{|\omega|=n,|\eta|\leq n,i,j,k,\nu} \frac{\partial P(\omega, j)}{\partial u(\eta, j)}(\phi, \alpha) D_{\eta+\delta(\nu)}\phi_j D_{\delta(k)}\alpha_\nu D_{\delta(i)}(\alpha^{-1})_k a_i u(\omega, j) \\
&\quad + \sum_{|\omega|=n,|\eta|\leq n,i,j,k,\nu} \frac{\partial P(\omega, j)}{\partial v(\eta, \nu)}(\phi, \alpha) D_{\eta+\delta(k)}\alpha_\nu D_{\delta(i)}(\alpha^{-1})_k a_i u(\omega, j) \\
&= (1) R_\alpha \sum_{|\eta|=n,i,j} D_{\eta+\delta(i)}\phi_j a_i u(\eta, j) \\
&\quad + (2) C_n R_\alpha \sum_{1\leq|\eta|\leq n-1,i,j} D_{\eta+\delta(i)}\phi_j a_i u(\eta, j) \\
&\quad + (3) C_n D(L_\phi)_\alpha \sum_{1\leq|\eta|\leq n,\nu,i,k} D_{\eta+\delta(k)}\alpha_\nu D_{\delta(i)}(\alpha^{-1})_k a_i v(\eta, \nu) \,.
\end{aligned}$$

Now $(2) = C_n R_\alpha(\mathrm{inj}) Dt_{\phi^{n-1}}(v)$ and $(3) = C_n D(L_\phi)_\alpha Dt_\alpha D\alpha^{-1}v$. Thus to demonstrate $(*)$ it must be shown that

$$([\phi\alpha]^n, C_n R_\alpha(\mathrm{inj}) Dt_{\phi^{n-1}}(v) + C_n D(L_\phi)_\alpha Dt_\alpha D\alpha^{-1}v) \in B \,.$$

But, by i), $(\mathrm{inj}\,(Dt_{\phi^{n-1}}(v)))_{[\phi]^n} \in TA$, so $(R_\alpha(\mathrm{inj})(Dt_{\phi^{n-1}}(v)))_{[\phi\alpha]^n} \in TA$. Thus, by ii), $([\phi\alpha]^n, C_n R_\alpha(\mathrm{inj}) Dt_{\phi^{n-1}}(v)) \in B$. Since $\tilde{\mathscr{L}}_p$ is translation invariant, $t_\alpha(x) \in \tilde{\mathscr{L}}_p$ for small $x \in \boldsymbol{R}^p$. Since A is invariant under $\tilde{\mathscr{L}}_p$, $L_\phi \circ t_\alpha(x) \in A$ for small x. It follows that $(D(L_\phi)_\alpha Dt_\alpha D\alpha^{-1}v)_{[\phi\alpha]^n} \in TA$. By ii), $([\phi\alpha]^n, C_n D(L_\phi)_\alpha Dt_\alpha D\alpha^{-1}v) \in B$, and hence the result.

Proposition 4.4. *Theorem 4.3 remains valid if $n = 1$, $\tilde{\mathscr{L}}_q = \{\mathrm{id}\}$, and condition ii) is replaced by ii)′: $B \supset \{([\phi]^1, [\psi]^1) \mid [\phi]^1 \in A$ and image $D\psi \subset$ image $D\phi\}$.*

Proof. A mild modification of the proof of Theorem 4.3.

Proposition 4.5. *Let $n \geq 1$ and let $\gamma\colon E_1 \to E_2^* \otimes E_3$ be as in Theorem 4.3. Suppose, in addition, that $B = \{([\phi]^n, [\psi]^n) \mid [\phi]^n \in A$, $[\psi]^n \in \tilde{J}^n(p, q)$ and*

$([\phi]^n)_{[\phi]^n} \in TA\}$. *If* $[\phi]^n \in A$, *let* $U(\phi) = \{v \in \boldsymbol{R}^p \mid ([\phi]^n, v) \in E_2$ *and* $Tt_{\phi^n}(v_0) \in TA\}$. *Then* $A_a(\gamma) = \{[\phi]^{n+1} \mid [\phi]^n \in A$ *and* $U(\phi)$ *is an a-dimensional vector space*$\}$.

Proof. Trivial.

Let γ be a-uniform. It follows from Proposition 4.1 that

$$S_a = \tilde{S}_a = \varepsilon_a\Big(G_a(\pi^* E_2) \times (\mathrm{id} \otimes \mu_n)^{-1}\Big(\bigcirc_{n+1} \boldsymbol{R}^{p*}\Big) \otimes \boldsymbol{R}^q\Big)\Big) .$$

Thus S_a is a factor bundle of $G_a(\pi^* E_2) \times \tilde{J}^{n+1}(p, q)$ and $s_a^* S_a$ is a factor bundle of $A_a(\gamma) \times \tilde{J}^{n+1}(p, q)$. It follows from Theorem 3.2 that there is an exact sequence $0 \to K_a^* \otimes N_a \to s_a^* S_a \to T(E_1, A_a(\gamma)) \to 0$. Thus $T(E_1, A_a(\gamma))$ is a factor bundle of $A_a(\gamma) \times \tilde{J}^{n+1}(p, q)$. In fact, if γ is equivariant, there is an exact sequence of H-bundles and equivariant maps $0 \to \tilde{B} \to A_a(\gamma) \times \tilde{J}^{n+1}(p, q) \to T(E_1, A_a(\gamma)) \to 0$ over $A_a(\gamma)$, where $\tilde{B} = \{(\phi, \psi) \in A_a(\gamma) \times \tilde{J}^{n+1}(p, q) \mid \psi_\phi \in TA_a(\gamma)\}$.

Note. Let $\gamma\colon E_1 \to E_2^* \otimes E_3$ be as in Theorem 4.3 with $n = 0$ or $B = \{(\phi, \psi) \in A \times \tilde{J}^n(p, q) \mid \psi_\phi \in TA\}$. Let $E = \{[\phi]^{n+2} \mid [\phi]^{n+1} \in A_a(\gamma)\}$ and let $\gamma'\colon E \to K_a^* \otimes T(E_1, A_a(\gamma))$ be the map induced by $H^{n+2}\colon E \to A_a(\gamma) \times (\boldsymbol{R}^{p*} \otimes \tilde{J}^{n+1}(p, q))$. Then γ' obeys the conditions of Theorem 4.3.

Suppose V and W are vector spaces and $\eta\colon V \to W$. Then η will be called a polynomial function if, relative to some choice of bases, each coordinate function of η is a polynomial in the coordinate functions of V. This condition does not depend on the choice of bases.

Let V and W be vector spaces, X a subset of V, and C a vector subbundle of $X \times W$. Suppose X is determined by polynomial equalities and inequalities. C will be called polynomially determined if there are an integer b and a polynomial $\eta\colon V \to \mathrm{Lin}\,(W, \boldsymbol{R}^b)$ such that $(x, w) \in C$ for $x \in X$ if and only if $\eta(x)(w) = 0$.

Proposition 4.6. *Let all notation be as in Theorem* 4.3. *Suppose* $E_2 \subset J^n(p, q) \times \boldsymbol{R}^p$ *and* $B \subset J^n(p, q) \times \tilde{J}^n(p, q)$ *are both polynomially determined. Then* $A_a(\gamma)$ *is determined by polynomial equalities and inequalities.*

Proof. Let $\sigma\colon J^n(p, q) \to \mathrm{Lin}\,(\boldsymbol{R}^p, \boldsymbol{R}^b)$ be a polynomial such that $([\phi]^n, v) \in E_2$ if and only if $[\phi]^n \in A$ and $\sigma([\phi]^n)(v) = 0$. Let $\tau\colon J^n(p, q) \to \mathrm{Lin}\,(\tilde{J}^n(p, q), \boldsymbol{R}^c)$ be a polynomial such that $([\phi]^n, [\psi]^n) \in B$ if and only if $[\phi]^n \in A$ and $\tau([\phi]^n)([\psi]^n) = 0$. Let $[\phi]^n \in A$. Then $[\phi]^{n+1} \in A_a(\gamma)$ if and only if

$$\Big\{(a_1, \cdots, a_p) \mid \sigma([\phi]^n)(a_1, \cdots, a_p) = 0 \text{ and } \tau([\phi]^n)\Big(\sum_{|\omega| = n, \nu, j} a_\nu D_{\omega + \delta(\nu)} \phi_j u(\omega)\Big) = 0\Big\}$$

is an a-dimensional vector space. Thus there is a polynomial $\eta\colon J^{n+1}(p, q) \to \mathrm{Lin}\,(\boldsymbol{R}^p, \boldsymbol{R}^{b+c})$ such that $[\phi]^{n+1} \in A_a(\gamma)$ if and only if $[\phi] \in A$ and $\eta([\phi]^{n+1})$ has rank $p - a$. Since determinant functions are polynomials, the result follows.

Proposition 4.7. *Assume the hypothesis of Proposition* 4.6. *Then* K_a *and* $\tilde{B}$ *are polynomially determined.*

Proof. Let η be the polynomial of the proof of Proposition 4.6. Then $([\phi]^{n+1}, v) \in K_a$ if and only if $[\phi]^{n+1} \in A_a(\gamma)$ and $\eta([\phi]^{n+1})(v) = 0$. We now show that $\tilde{B}$ is polynomially determined. If $\phi \in A_a(\gamma)$, let $B_\phi = \{\psi \in \tilde{J}^n(p,q) \mid ([\phi]^n, \psi) \in B\}$ and $F_\phi = \{w \in \boldsymbol{R}^{p^*} \mid w(v) = 0$ whenever $(\phi, v) \in K_a\}$. Let

$$C_\phi = \mu_{n+1}^{-1}(((\mathrm{id} \otimes \mu_n)(\boldsymbol{R}^{p^*} \otimes B_\phi + F_\phi \otimes \tilde{J}^n(p, q))) \cap \left(\left(\bigcirc_{n+1} \boldsymbol{R}^{p^*}\right) \otimes \boldsymbol{R}^q\right)\right) .$$

Let $C = \{(\phi, \psi) \mid \phi \in A_a(\gamma)$ and $\psi \in C_\phi\}$. The bundle C is polynomially determined. It follows from Proposition 4.1 and the exactness of $0 \to B \to A \times \tilde{J}^n(p, q) \to E_3 \to 0$ that there is an exact sequence $0 \to C \to A \times \tilde{J}^{n+1}(p, q) \to s_a^* S_a \to 0$.

If $\phi \in A_a(\gamma)$, let

$$P_\phi = \left\{\sum_{|\omega|=n, \nu, j} a_\nu D_{\omega+\delta(\nu)} \phi_j u(\omega, j) \mid ([\phi]^n, (a_1, \cdots, a_p)) \in E_2\right\} .$$

Each P_ϕ may be described in terms of polynomials in the coordinates of ϕ. Since $0 \to K_a^* \otimes N_a \to s_a^* S_a \to T(E_1, A_a(\gamma)) \to 0$ is exact, so is $K_a^* \otimes s_a^* S_a \to T(E_1, A_a(\gamma)) \to 0$. It follows that

$$\tilde{B} = \left\{(\phi, \psi) \mid \phi \in A_a(\gamma),\ \psi \in C_\phi + \mu_{n+1}^{-1}\left(((\mathrm{id} \otimes \mu_n)(\boldsymbol{R}^{p^*} \otimes P_\phi)) \cap \left(\left(\bigcirc_{n+1} \boldsymbol{R}^{q^*} \otimes \boldsymbol{R}^q\right)\right)\right)\right\}$$

and is therefore polynomially determined.

5. Singularities of mappings

Let V be a manifold of type G, and suppose G acts on F. $\underline{F}$ will denote the bundle with base V, fiber F and group G. If U is a subset of F, which is invariant under G, then $\underline{U}$ is a sub-bundle of $\underline{F}$. Let W be a bundle over U, and suppose G acts on W in such a way that the bundle projection $W \to U$ is equivariant. Then W induces a bundle $\underline{W}$ over $\underline{U}$ with group G and fiber that of W. Suppose G acts on bundles W_1 and W_2 over U in such a way that the bundle projections are equivariant. If $\phi\colon W_1 \to W_2$ is an invariant bundle morphism, then ϕ induces a morphism $\underline{\phi}\colon \underline{W}_1 \to \underline{W}_2$. If W_1 and W_2 are G-bundles and $\phi\colon W_1 \to W_2$ is an equivariant morphism of vector bundles, then $\underline{\phi}$ is a morphism of vector bundles. Furthermore, $\_$ takes commutative diagrams into commutative diagrams and exact sequences into exact sequences.

Let all notation be as in § 4, and $\tilde{\mathscr{L}}_p$ and $\tilde{\mathscr{L}}_q$ translation invariant subgroups of $\mathscr{L}_p$ and $\mathscr{L}_q$ respectively. Suppose $\gamma\colon E_1 \to E_2^* \otimes E_3$ is a-uniform and satisfies the hypotheses of either Theorem 4.3 or Proposition 4.4 (so γ is equivariant). Let X be a manifold of type $\tilde{\mathscr{L}}_p$ and Y a manifold of type $\tilde{\mathscr{L}}_q$. It follows from Corollary 3.5 that over $J^{n+1}_{A_a(\gamma)}(X, Y)$ there is an exact sequence

$$0 \to \underline{K_a^* \otimes N_a} \to \underline{s_a^* S_a} \to T(J^{n+1}_{E_1}(X, Y), J^{n+1}_{A_a(\gamma)}(X, Y)) \to 0 .$$

Note also that $\underline{K_a^* \otimes N_a} \approx \underline{(K_a)^* \otimes N_a}$. In the furture, underlines will be dropped, $\underline{s_a^* S_a}$ will be abbreviated to F_a, and $T(J_{E_1}^{n+1}(X, Y), J_{A_a(\gamma)}^{n+1}(X, Y))$ to R_a. Thus the above sequence becomes $0 \to K_a^* \otimes N_a \to F_a \to R_a \to 0$ over $J_{A_a(\gamma)}^{n+1}(X, Y)$.

Let $r_n: J^n(X, Y) \to J^{n-1}(X, Y)$, $r^n: J^n(X, Y) \to X \times Y$ for $n \geq 1$, and $\varepsilon_1: X \times Y \to X$ and $\varepsilon_2: X \times Y \to Y$ be the projections. For $n \geq 1$, define $\tilde{J}^n(X, Y) = \{\phi \in J^n(X, Y) \mid \phi \text{ is } (n-1)\text{-equivalent to a constant germ}\}$. $\tilde{J}^n(X, Y)$ is a vector bundle over $X \times Y$ and, in fact, $\tilde{J}^n(X, Y) \approx \left(\bigodot_n \varepsilon_1^* TX^*\right) \otimes \varepsilon_2^* TY$ $\approx \underline{\{0\} \times \tilde{J}^n(p, q)}$.

E_3 is a factor bundle of $r^{n*}\tilde{J}^n(X, Y)$ over $J_A^n(X, Y)$ because of the exactness of $0 \to B \to A \times \tilde{J}^n(p, q) \to E_3 \to 0$ over A. Thus if $n = 1$, then E_3 is a factor bundle of $TJ^1(X, Y) = Tr_1^{-1}Tr_1J_A^1(X, Y)$ over $J_A^1(X, Y)$. Note that for $n \geq 1$ there is an exact sequence $0 \to r^{n*}\tilde{J}^n(X, Y) \to TJ^n(X, Y) \to r_n^* TJ^{n-1}(X, Y) \to 0$, so B is a sub-bundle of $TJ^n(X, Y)$ over $J_A^n(X, Y)$. If $n \geq 2$, it follows from the hypotheses of Theorem 4.3 that there is an exact sequence $0 \to TJ_A^n(X, Y) + B \to Tr_n^{-1}Tr_nTJ_A^n(X, Y) \to E_3 \to 0$. Therefore for $n \geq 1$ there is an epimorphism $\varepsilon: Tr_n^{-1}Tr_nTJ_A^n(X, Y) \to E_3$. If $n = 0$, then E_3 is a factor bundle of TY over Y. The epimorphism $TY \to E_3$ will also be denoted ε.

Let $f: X \to Y$. $A_a(\gamma)(f)$ will be abbreviated to $A_a(f)$. Note finally that if $n \geq 1$, then $f^{n*}E_2$ is a sub-bundle of TX over $A(f)$. In the case $n = 0$, E_2 is a subbundle of TX.

Proposition 5.1. *Let $n = 0$ and $f: X \to Y$. Then $A_a(f) = \{x \in X \mid$ dimension kernel $(\varepsilon \circ Tf) \mid (E_2)_x = a\}$.*

Proof. Trivial.

Proposition 5.2. *Let $n \geq 1$ and $f: X \to Y$. Then $A_a(f) = \{x \in A(f) \mid$ dimension kernel $(\varepsilon \circ Tf^n) \mid (f^{n*}E_2)_x = a\}$.*

Proof. This is a local question. Assume $X = \boldsymbol{R}^p$, $Y = \boldsymbol{R}^q$, $x = 0$, $f(0) = 0$, and $0 \in A(f)$. $J^n(\boldsymbol{R}^p, \boldsymbol{R}^q) = \boldsymbol{R}^n \times \boldsymbol{R}^q \times J^n(p, q)$. Let $\bar{f}^n$ be the projection of f^n on $J^n(p, q)$. $T\bar{f}^n(v_0) = (Dt_{f^n}(v))_{[f]^n}$. Let $v_0 \in f^{n*}E_2$, implying $([f]^n, v) \in E_2$ so $((\text{inj})Dt_{f^{n-1}}(v))_{[f]^n} \in TA$. It follows that for $v = (a_1, \cdots, a_p)$, $(\varepsilon \circ Tf^n)(v_0) = 0$ if and only if $\left([f]^n, \sum_{|\omega| = \hat{n}, \nu, j} a_\nu D_{\omega + \delta(\nu)} f_j u(\omega, j)\right) \in B$. Thus $0 \in A_a(f)$ if and only if kernel $(\varepsilon \circ Tf^n)/(f^{n*}E_2)_0$ has dimension a. q.e.d.

R_a is a factor bundle of $r^{n+1*}\tilde{J}^{n+1}(X, Y)$ over $J_{A_a(\gamma)}^{n+1}(X, Y)$. Thus, if $f: X \to Y$, then $f^{n+1*}R_a$ is a factor bundle of $\left(\bigodot_{n+1} TX^*\right) \otimes f^*TY$.

Suppose f is A-transversal; so $A(f)$ is a manifold. $Tf^n(TA(f)) \subset TJ_A^n(X, Y)$ so $Tf^{n+1}(TA(f)) \subset Tr_{n+1}^{-1}TJ_A^n(X, Y) = TJ_{E_1}^{n+1}(X, Y)$. Since there is a map $TJ_{E_1}^{n+1}(X, Y) \to R_a$ over $A_a(\gamma)$, Tf^{n+1} induces a map $TA(f) \to R_a$ over $A_a(f)$ and hence a map $\phi: TA(f) \to f^{n+1*}R_a$ over $A_a(f)$.

Since f is A-transversal, Tf^{n+1} induces an exact commutative diagram

$$\begin{array}{ccccccc} 0 \to TA(f) & \longrightarrow & TX & \longrightarrow & T(X, A(f)) & \longrightarrow & 0 \\ \downarrow \phi & & \downarrow \eta & & \wr\wr & & \\ 0 \to f^{n+1*}R_a \to & f^{n+1*}T(J^{n+1}(X, Y), J^{n+1}_{A_a(\gamma)}(X, Y)) & \to & f^{n*}T(J^n(X, Y), J^n_A(X, Y)) & \to 0 \end{array}$$

over $A_a(f)$. f is $A_a(\gamma)$-transversal if and only if η is an epimorphism if and only if ϕ is an epimorphism. Hence we have shown

Proposition 5.3. *Let $f\colon X \to Y$. Then $f^{n+1*}R_a$ is a factor bundle of $\left(\bigcirc_{n+1} TX^*\right) \otimes f^*TY$ over $A_a(f)$. If f is A-transversal, then Tf^{n+1} induces a map $TA(f) \to f^{n+1*}R_a$ over $A_a(f)$. f is $A_a(\gamma)$-transversal if and only if this map is an epimorphism.*

Let f be $A_a(\gamma)$-transversal, $x \in A(f)$ and $v \in TX_x$. Then $v \in TA_a(f)$ if and only if $Tf^{n+1}(v) \in TJ^{n+1}_{A_a(\gamma)}(X, Y)$. Thus

Proposition 5.4. *Let f be $A_a(\gamma)$-transversal. Then, over $A_a(f)$, $TA_a(f)$ is the kernel of $TA(f) \to {}^{n+1*}R_a$.*

6. Examples and applications

Let V be a vector bundle over X, and suppose W_1 is a factor bundle of $\bigcirc_m V$ and W_2 is a factor bundle of $\bigcirc_n V$. Then $W_1 \otimes W_2$ is a factor bundle of $\left(\bigcirc_m V\right) \otimes \left(\bigcirc_n V\right) \to W_1 \otimes W_2$. Define $W_1 \circ W_2$ to be the image of $\bigcirc_{m+n} V$. Since the fiber dimension of $W_1 \circ W_2$ may vary from point to point of X, $W_1 \circ W_2$ is not necessarily a bundle.

If W_1 is a factor bundle of $X \times \left(\bigcirc_m \boldsymbol{R}^{p*}\right)$ and W_2 is a factor bundle of $X \times \tilde{J}^n(p, q)$, then $W_1 \otimes W_2$ is a factor bundle of $X \times \left(\left(\bigcirc_m \boldsymbol{R}^{p*}\right) \otimes \tilde{J}^n(p, q)\right) = X \times \left(\left(\bigcirc_m \boldsymbol{R}^{p*}\right) \otimes \left(\bigcirc_n \boldsymbol{R}^{p*}\right) \otimes \boldsymbol{R}^q\right)$. Define $W_1 \circ W_2$ to be the image of $X \times \left(\left(\bigcirc_{m+n} \boldsymbol{R}^{p*}\right) \otimes \boldsymbol{R}^q\right) = X \times \tilde{J}^{m+n}(p, q)$. Once again, $W_1 \circ W_2$ need not be a bundle.

Consideration of the special case, where X is a point, yields similar definitions for the symmetric product of appropriate vector spaces.

Let W_1, W_2 and W_3 be factor bundles of $X \times \left(\bigcirc_k V\right)$, $X \times \left(\bigcirc_m V\right)$, and $X \times \left(\bigcirc_n V\right)$ respectively, and suppose $W_1 \circ W_2$ and $W_2 \circ W_3$ are bundles. Then $W_1 \circ (W_2 \circ W_3) = (W_1 \circ W_2) \circ W_3$, so parentheses may be removed without introducing ambiguity. Similarly, if W_1, W_2 and W_3 are factor bundles of $X \times \left(\bigcirc_k \boldsymbol{R}^{p*}\right)$, $X \times \left(\bigcirc_m \boldsymbol{R}^{p*}\right)$, and $X \times \tilde{J}^n(p, q)$ respectively.

If $0 \leq p \leq q$, there is an epimorphism $\boldsymbol{R}^q \to \boldsymbol{R}^p$ defined by $(x_1, \cdots, x_q) \to$

$(x_1, \cdots, x_p)$. Suppose $I^m = (a_1, \cdots, a_m)$ is such that each a_i is a non-negative integer and $a_1 \geq \cdots \geq a_m$. Since each of the vector spaces R^{a_i} is a factor space of R^{a_1}, $R^{a_m} \circ \cdots \circ R^{a_1}$ is defined. Define $P(I^m) =$ dimension $(R^{a_m} \circ \cdots \circ R^{a_1})$.

Lemma 6.1. *Let $W_1, \cdots, W_m$ be vector bundles over X, and suppose that for each i there is an epimorphism $W_i \to W_{i+1}$. Then $W_m \circ \cdots \circ W_1$ is a vector bundle. If, for each i,* $\dim W_i = a_i$, *then* $\dim (W_m \circ \cdots \circ W_1) = P(a_1, \cdots, a_m)$.

Proof. Straightforward.

Let p and q be given. Define an admissible sequence of length n to be a tuple $(a_1, \cdots, a_n)$ of non-negative integers such that $a_1 \geq p - q$ and $p \geq a_1 \geq \cdots \geq a_n$. If $I^n = (a_1, \cdots, a_n)$ is an admissible sequence of length n and $0 \leq m \leq n$, then $I^m = (a_1, \cdots, a_m)$ is an admissible sequence of length m.

Fix an admissible sequence $I^n = (a_1, \cdots, a_n)$. If $0 \leq i \leq j$, let $r_i^j : J^j(p, q) \to J^i(p, q)$ be the projection.

Define $Z(\phi) = \{0\} = J^0(p, q)$. Let $E_2^0 = Z(\phi) \times R^p$ and $E_3^0 = Z(\phi) \times R^q$. Now suppose that whenever $1 \leq m \leq n - 1$, $Z(I^m)$ is a submanifold of $J^m(p, q)$. If $1 \leq m \leq n$, let $E_1^m = \{\phi \in J^m(p, q) \mid [\phi]^{m-1} \in Z(I^{m-1})\}$. If $1 \leq m \leq n - 1$, let $B^m = \{(\phi, \psi) \in Z(I^m) \times \tilde{J}^m(p, q) \mid \psi_\phi \in TZ(I^m)\}$ and assume it to be a bundle over $Z(I^m)$. If $1 \leq m \leq n - 1$, define E_3^m over $Z(I^m)$ by the exactness of $0 \to B^m \to Z(I^m) \times \tilde{J}^m(p, q) \to E_3^m \to 0$. If $0 \leq m \leq n - 1$, let E_2^m be a vector subbundle of $Z(I^m) \times R^p$. If $0 \leq m \leq n-1$, $H^{m+1} : E_1^{m+1} \to Z(I^m) \times (R^{p*} \otimes \tilde{J}^m(p, q))$ induces $\gamma^{m+1} : E_1^{m+1} \to E_2^{m*} \otimes E_3^m$. If $0 \leq m \leq n - 2$, suppose γ^{m+1} is a_{m+1}-uniform and $Z(I^{m+1}) = Z(I^m)_{a_{m+1}}(\gamma^{m+1})$. Define $Z(I^n) = Z(I^{n-1})_{a_n}(\gamma^n)$. If $0 \leq m \leq n - 1$, γ^{m+1} induces a map $r_m^{m+1*}E_2^m \to r_m^{m+1*}E_3^m$ over E_1^{m+1}. If $0 \leq m \leq n - 2$, suppose this map induces an exact sequence $0 \to E_2^{m+1} \to r_m^{m+1*}E_2^m \to r_m^{m+1*}E_3^m \to Q^{m+1} \to 0$ over $Z(I^{m+1})$ defining Q^{m+1}. (Note that the bundles E_2^m and the sets $Z(I^m)$ are defined inductively.) Define bundles E_2^n and Q^n over $Z(I^n)$ by the exactness of $0 \to E_2^n \to r_{n-1}^n{}^*E_2^{n-1} \to r_{n-1}^n{}^*E_3^{n-1} \to Q^n \to 0$. If $1 \leq m \leq n$, define a bundle N^m over $Z(I^m)$ by the exactness of $0 \to E_2^m \to r_{m-1}^m{}^* \to E_2^{m-1} \to N^m \to 0$.

Let $\pi : G_{a_n}(r_{n-1}^n{}^*E_2^{n-1}) \to E_1^n$ be the bundle projection, and $0 \to L_{a_n} \to \pi^* r_{n-1}^n{}^*E_2^{n-1} \to M_{a_n} \to 0$ the usual sequence as in § 2. If $s^n : Z(I^n) \to G_{a_n}(r_{n-1}^n{}^*E_2^{n-1})$ is the standard section, then $s^{n*}L_{a_n} = E_2^n$ and $s^{n*}M_{a_n} = N^n$.

If $1 \leq i \leq n - 1$, $\gamma^i : E_1^i \to E_2^{i-1*} \otimes E_3^{i-1}$ over $Z(I^{i-1})$ induces a monomorphism $N^i \to r_{i-1}^i{}^*E_3^{i-1}$ and hence, over $G_{a_n}(r_{n-1}^n{}^*E_2^{n-1})$, a monomorphism

$$\begin{aligned} L_{a_n}^* \circ \pi^*(r_{n-1}^n{}^*E_2^{n-1*} \circ \cdots \circ r_i^{n*}E_2^{i*}) \otimes \pi^* r_i^{n*} N_i \\ \to L_{a_n}^* \circ \pi^*(r_{n-1}^n{}^*E_2^{n-1*} \circ \cdots \circ r_i^{n*}E_2^{i*}) \otimes \pi^* r_{i-1}^n{}^*E_3^{i-1} . \end{aligned}$$

It is annoying but straightforward to show that the image of this map is contained in the symmetric subset $L_{a_n}^* \circ \pi^*(r_{n-1}^n{}^*E_2^{n-1*} \circ \cdots \circ r_i^{n*}E_{2*}^i \circ r_{i-1}^n{}^*E_3^{i-1})$. $0 \to N^i \to r_{i-1}^i{}^*E_3^{i-1} \to Q^i \to 0$ and $0 \to E_2^{i*} \otimes N^i \to E_2^{i*} \circ r_{i-1}^i{}^*E_3^{i-1} \to E_2^{i*} \circ Q^i \to 0$ are exact. But for $1 \leq i \leq n - 1$, $E_2^{i*} \circ Q^i \approx E_3^i$ by Proposition 4.1 and Theorem 3.2. Thus over each point of $G_{a_n}(r_{n-1}^n{}^*E_2^{n-1})$ there are exact sequences:

$$0 \to L^*_{a_n} \circ \pi^* r^{n*}_{n-1} E_2^{n-1*} \otimes \pi^* r^{n*}_{n-1} N^{n-1}$$
$$\to L^*_{a_n} \circ \pi^* r^{n*}_{n-1} E_2^{n-1*} \circ \pi^* r^{n*}_{n-2} E_3^{n-2} \to L^*_{a_n} \circ \pi^* r^{n*}_{n-1} E_3^{n-1} \to 0 ,$$

$$0 \to L^*_{a_n} \circ \pi^* r^{n*}_{n-1} E_2^{n-1*} \circ \pi^* r^{n*}_{n-2} E_2^{n-2*} \otimes \pi^* r^{n*}_{n-2} N^{n-2}$$
$$\to L^*_{a_n} \circ \pi^* r^{n*}_{n-1} E_2^{n-1*} \circ \pi^* r^{n*}_{n-2} E_2^{n-2*} \circ \pi^* r^{n*}_{n-3} E_3^{n-3}$$
$$\to L^*_{a_n} \circ \pi^* r^{n*}_{n-1} E_2^{n-1*} \circ \pi^* r^{n}_{n-2} E_3^{n-2} \to 0 ,$$

. . . .

$$0 \to L^*_{a_n} \circ \pi^* r^{n*}_{n-1} E_2^{n-1*} \circ \cdots \circ \pi^* r_1^{n*} E_2^{1*} \otimes \pi^* r_1^{n*} N^1$$
$$\to L^*_{a_n} \circ \pi^* r^{n*}_{n-1} E_2^{n-1*} \circ \cdots \circ \pi^* r_1^{n*} E_2^{1*} \otimes \pi^* r_0^{n*}(\{0\} \times \boldsymbol{R}^q)$$
$$\to L^*_{a_n} \circ \pi^* r^{n*}_{n-1} E_2^{n-1*} \circ \cdots \circ \pi^* r_2^{n*} E_2^{2*} \circ \pi^* r_1^{n*} E_3^1 \to 0 .$$

Note that the fiber dimension of N^1 is $p - a_1$, and the fiber dimension of N^i is $a_{i-1} - a_i$ for $i > 1$. Thus from Lemma 6.1 and the exactness of the above sequences, the fiber dimension of $L^*_{a_n} \circ \pi^* r^{n*}_{n-1} E_3^{n-1}$ at each point of $G_{a_n}(r^{n*}_{n-1} E_2^{n-1})$ is $P(a_1, \cdots, a_n)(q - p + a_1) - \sum_{i=2}^{n-1} P(a_i, \cdots, a_n)(a_{i-1} - a_i)$. Consequently, γ^n is a_n-uniform and therefore $Z(I^n)$ is a manifold. Furthermore, $0 \to E_2^{n*} \otimes N^n \to E_2^{n*} \circ r^{n*}_{n-1} E_3^{n-1} \to T(E_1^n, Z(I^n)) \to 0$ is exact. Thus $T(E_1^n, Z(I^n)) \approx E_2^{n*} \circ Q^n$ and has dimension $P(a_1, \cdots, a_n)(q - p + a_1) - \sum_{i=2}^{n} P(a_i, \cdots, a_n)$ $(a_{i-1} - a_i)$.

That $Z(I^n)$ is invariant under $\mathscr{L}_p \times \mathscr{L}_q$ is immediate from Theorem 4.3. If U and V are manifolds and ϕ: $U \to V$, let $Z_a(\phi) = Z(a)(\phi) = \{x \in U \mid \text{dimension kernel } T\phi_x = a\}$. Let X be a p-manifold, Y a q-manifold, and let f: $X \to Y$ be $Z(I^m)$-transversal for each $m \leq n - 1$. It follows from Proposition 5.4 that, for each m, $Z(I^{m+1})(f) = Z_{a_{m+1}}(f / Z(I^m)(f))$.

We now summarize:

Theorem 6.2 (*Boardman*). *Let X be a (compact) p-manifold, Y a q-manifold and $I^n = (a_1, \cdots, a_n)$ an admissible sequence. If f: $X \to Y$, define $Z(\phi)(f) = X$, and if $Z(I^m)(f) \subset X$ is defined and is a manifold, define $Z(I^{m+1})(f) = Z_{a_{m+1}}(f / Z(I^m)(f))$. Then for a (open and dense) dense set of functions f in $\mathscr{C}^{n+1}(X, Y)$, $Z(I^m)(f)$ is a manifold for $1 \leq m \leq n$, and furthermore, for such f,*

$$\text{dimension } T(Z(I^{n-1})(f), Z(I^n(f))) = P(I^n)(q - p + a_1) - \sum_{i=1}^{n} P(a_i, \cdots, a_n)(a_{i-1} - a_i) .$$

Proposition 6.3. *Let $\tilde{\mathscr{L}}_p = \{\alpha_0 \mid \alpha \in \mathfrak{F}(p, p), \alpha_0 \in \mathscr{L}_p$ and, for sufficiently small x, $D\alpha_x$ preserves perpendicularity$\}$. Let $a > p - q$ and* max $(0, a(q - p + a) + a - p) \leq b \leq a$, *then there is a submanifold $Z(a \perp b)$ of $(r_1^2)^{-1} Z(a)$, invariant under $\tilde{\mathscr{L}}_p \times \mathscr{L}_q$, such that:*

i) *dimension $T((r_1^2)^{-1} Z(a), Z(a \perp b)) = b(p - a(q - p + a) - (a - b))$,*

ii) *if X is a manifold of type $\tilde{\mathscr{L}}_p$, Y is a q-manifold and $f\colon X \to Y$ is $Z(a)$-transversal, then $Z(a \perp b)(f) = \{x \in Z(a)(f) \mid$ the intersection of the vector space normal to $TZ(a)(f)_x$ with kernel Tf_x is b-dimensional$\}$.*

Proof. Over $Z(a)$, H^1 induces an exact sequence $0 \to K_a \to Z(a) \times \boldsymbol{R}^p \to Z(a) \times \boldsymbol{R}^q \to Q_a \to 0$. Furthermore, $T(J^1(p, q), Z(a)) \approx K_a^* \otimes Q_a$. Define E over $Z(a)$ by $E = \{(\phi, v) \in Z(a) \times \boldsymbol{R}^p \mid v$ is perpendicular to w whenever $(\phi, w) \in K_a\}$, E is an $\tilde{\mathscr{L}}_p \times \mathscr{L}_q$ bundle over $Z(a)$ with fiber dimension $p - a$. H^2 induces $\gamma^2\colon (r_1^2)^{-1}Z(a) \to E^* \otimes K_a^* \otimes Q_a$. Define $Z(a \perp b) = Z(a)_{p-a(q-p+a)-(a-b)}(\gamma^2)$. γ^2 is $p - a(q - p + a) - (a - b)$ uniform sice $E \cap K_a$ is the zero section of $Z(a) \times \boldsymbol{R}^p$. Over $Z(a \perp b)$, γ^2 induces and exact sequence $0 \to K_{a\perp b} \to r_1^{2*}E \to r_1^{2*}(K_a^* \otimes Q_a)$ where dimension $(K_{a\perp b}) = p - a(q - p + a) - (a - b)$. If $N_{a\perp b}$ is defined by the exactness of $0 \to K_{a\perp b} \to r_1^{2*}E \to N_{a\perp b} \to 0$, there is an exact sequence $0 \to K_{a\perp b}^* \otimes N_{a\perp b} \to K_{a\perp b}^* \otimes r_1^{2*}(K_a^* \otimes Q_a) \to T((r_1^2)^{-1}Z(a), Z(a \perp b)) \to 0$. That $Z(a \perp b)$ is invariant under $\tilde{\mathscr{L}}_p \times \mathscr{L}_q$ is immediate from Theorem 4.3. It remains to show ii).

Let X be a manifold of type $\tilde{\mathscr{L}}_p$ and Y a q-manifold. Let $f\colon X \to Y$ be $Z(a)$-transversal and let $x \in Z(a)(f)$. By Proposition 5.4, $x \in Z(a \perp b)$ if and only if $(f^{1*}E)_x \cap (TZ(a)(f))_x$ has dimension $p - a(q - p + a) - (a - b)$. But

$$((f^{1*}E)_x \cup TZ(a)(f)_x)^{\perp} = (f^{1*}E)_x^{\perp} + TZ(a)(f)_x^{\perp} = (f^{1*}K_a)_x + TZ(a)(f)_x^{\perp} .$$

Thus $x \in Z(a \perp b)$ if and only if

$$\begin{aligned} &a(q - p + a) + (a - b) \\ &\quad = \dim ((f^{1*}E)_x \cap TZ(a)(f)_x)^{\perp} = \dim ((f^{1*}K_a)_x + TZ(a)(f)_x^{\perp}) \\ &\quad = \dim f^{1*}K_a + \dim TZ(a)(f)^{\perp} - \dim ((f^{1*}K_a)_x \cap TZ(a)(f)_x^{\perp}) \\ &\quad = a + a(q - p + a) - \dim ((f^{1*}K_a)_x \cap TZ(a)(f)_x^{\perp}) \end{aligned}$$

if and only if $\dim ((f^{1*}K_a)_x \cap TZ(a)(f)_x^{\perp}) = b$. q.e.d.

Obviously Proposition 6.3 is not the most general result possible. One can construct invariant manifolds by combining perpendicularity considerations with the constructions of Theorem 6.2.

Proposition 6.4. *Let $\tilde{\mathscr{L}}_p \subset \mathscr{L}_p$ be translation invariant, $\tilde{\mathscr{L}}_q = \{\mathrm{id}\}$, and Q_a be as in the proof of Proposition 6.3. Let E be a vector sub-bundle of $Z(a) \times \boldsymbol{R}^p$ invariant under the action of $\tilde{\mathscr{L}}_p$, and $\gamma^2\colon (r_1^2)^{-1}Z(a) \to E^* \otimes (Z(a) \times \boldsymbol{R}^{p*}) \otimes Q_a$ be the map induced by H^2. If $b \leq \dim E$, then $Z(a)_b(\gamma^2)$ is a manifold and is invariant under $\tilde{\mathscr{L}}_p$.*

Proof. γ^2 is b-uniform by Lemma 6.1, so $Z(a)_b(\gamma^2)$ is a manifold. γ^2 is equivariant by Proposition 4.4. so $Z(a)_b(\gamma^2)$ is invariant under $\tilde{\mathscr{L}}_p$. q.e.d.

We conclude this section with an application of Proposition 6.4.

Let X be a p-manifold, and $f\colon X \to \boldsymbol{R}^q$ an immersion. f induces a map $\tilde{f}\colon X \to G_p(\boldsymbol{R}_q)$ defined by $Tf(TX_x) = (\tilde{f}(x))_{f(x)}$. According to Proposition 2.2,

$\bar{f}^*TG_p(\boldsymbol{R}^q) \approx TX^* \otimes f^{1*}Q_0$. Thus $T\bar{f}$ induces a map ϕ: $TX \to TX^* \otimes f^{1*}Q_0$. If, in Proposition 6.4, $a = 0$ and $E = Z(0) \times \boldsymbol{R}^p$, then a straightforward local analysis shows that $\phi = f^{2*}\gamma^2$. It follows from Proposition 6.4 that for $b \leq p$ and f suitably transversal, $Z_b(\bar{f})$ is a submanifold of X. Define bundles K_b^2 and N_b^2 over $Z_b(\bar{f})$ by the exactness of the sequences $0 \to K_b^2 \to TX \to TX^* \otimes f^{1*}Q_0$ and $0 \to K_b^2 \to TX \to N_b^2 \to 0$. For $Z(0)_b(\gamma^2)$-transversal immersions f there is an exact sequence $0 \to K_b^{2*} \otimes N_b^2 \to K_b^{2*} \bigcirc TX^* \otimes f^{1*}Q_0 \to T(X, Z_b(\bar{f})) \to 0$ over $Z_b(\bar{f})$. Thus $T(X, Z_b(\bar{f}))$ has dimension $(\frac{1}{2}b(b+1) + b(p-b))(q-p) - b(p-b) = \frac{1}{2}b(b+1)(q-p) + b(p-b)(q-p-1)$.

Proposition 6.5. *Let X be a compact p-manifold and let $q \geq p + 2$. Then there is a set $\mathscr{S}$ of immersions of X into $\boldsymbol{R}^q$, which is open and dense in the set of all immersions of X into $\boldsymbol{R}^q$ (in $\mathscr{C}^2(X, \boldsymbol{R}^q)$) such that $\bar{f}$ is an immersion for each $f \in \mathscr{S}$.*

Proof. If $b \geq 1$ and $q \geq p + 2$, then $\frac{1}{2}b(b+1)(q-p) + b(p-b)(q-p-1) \geq b(b+1) + b(p-b) = b(p+1) > p$.

7. Characteristic classes

In this section it will be shown that there is a connection between certain kinds of singularities of nice maps of manifolds and the Whitney classes of the domain and target manifolds. Since the results are fragmentary, only a sketch of the methodology will be given. The approach was outlined by Porteous in [5].

Let $\tilde{\mathscr{L}}_p$ (respectively $\tilde{\mathscr{L}}_q$) be a subgroup of $\mathscr{L}_p$ (respectively $\mathscr{L}_q$), and $A \subset J^n(p, q)$ a manifold invariant under $\tilde{\mathscr{L}}_p \times \tilde{\mathscr{L}}_q$. Let $E_1 = \{[\phi]^{n+1} \mid [\phi]^n \in A\}$, and let π: $E_1 \to A$ be the bundle projection. Let E_2 be a vector sub-bundle of $A \times \boldsymbol{R}^p$, which is invariant under $\tilde{\mathscr{L}}_p \times \tilde{\mathscr{L}}_q$, and let $0 \to B \to A \times \bar{J}^n(p, q) \to E_3 \to 0$ be an exact sequence over A with B invariant under $\tilde{\mathscr{L}}_p \times \tilde{\mathscr{L}}_q$. Let γ: $E_1 \to E_2^* \otimes E_3$ be the map induced by H^{n+1}, and suppose that γ is equivariant and a-uniform ($a \leq$ fiber dimension E_2). Let X be a manifold of type $\tilde{\mathscr{L}}_p$, and Y a manifold of type $\tilde{\mathscr{L}}_q$.

Then, as in § 5, $J_A^n(X, Y)$ and $J_{A_a(\gamma)}^{n+1}(X, Y)$ are manifolds, and E_2 and E_3 determine bundles (also denoted E_2 and E_3) over $J_A^n(X, Y)$. Also γ induces a map γ: $J_{E_1}^n(X, Y) \to E_2^* \otimes E_3$ over $J_A^n(X, Y)$, and we have a bundle $G_a(\pi^*E_2)$ over $J_{E_1}^{n+1}(X, Y)$ and an exact sequence $0 \to L_a \to \pi^*E_2 \to M_a \to 0$ over $G_a(\pi^*E_2)$ where $\bar{\pi}$: $G_a(\pi^*E_2) \to J_{E_1}^{n+1}(X, Y)$ is the bundle projection. Let γ_a: $G_a(\pi^*E_2) \to L_a^* \otimes \bar{\pi}^*\pi^*E_3$ be the section induced by γ. Since γ is a-uniform, there is a symmetric sub-bundle S_a of $L_a^* \otimes \bar{\pi}^*\pi^*E_3$, containing the image of γ_a, such that γ_a is a transversal section of S_a.

Let f: $X \to Y$. f^{n+1} induces a map $\bar{f}$: $G_a(f^{n*}E_2) \to G_a(\pi^*E_2)$. If $\tilde{\pi}$: $G_a(f^{n*}E_2) \to A(f)$ is the bundle projection, and $0 \to \tilde{L}_a \to \tilde{\pi}^*f^{n*}E_2 \to \tilde{M}_a \to 0$ is the obvious sequence over $G_a(f^{n*}E_2)$, then $\tilde{L}_a = \bar{f}^*L_a$ and $\tilde{M}_a = \bar{f}^*M_a$. γ: $E_1 \to E_2^* \otimes E_3$ induces a vector bundle morphism $\tilde{\gamma}$: $f^{n*}E_2 \to f^{n*}E_3$ which, in turn, induces a section $\tilde{\gamma}_a\colon G_a(f^{n*}E_2) \to \tilde{L}_a^* \otimes \bar{\pi}^* f^{n*}E_3$. Since $\tilde{\gamma}_a$ is the pullback $\bar{f}^*\gamma_a$ of the section γ_a, the image of $\tilde{\gamma}_a$ is contained in the symmetric sub-bundle $\bar{f}^*S_a$. Note that $A_a(f) = \{x \in A(f) \mid \text{dimension kernel } \tilde{\gamma}_x = a\}$. Define a section $\tilde{s}_a\colon A_a(f) \to G_a(f^{n*}E_2)$ by $\tilde{s}_a(x) = \text{kernel } \tilde{\gamma}_x$. Suppose f is A-transversal. It is not difficult to show that f is $A_a(\gamma)$-transversal if and only if $\tilde{\gamma}_a$ is a transversal section of $\bar{f}^*S_a$ on $\tilde{s}_a A_a(f)$.

If U is a topological space, then $H_*(U)(H^*(U))$ will denote the singular homology (cohomology) of U with Z_2-coefficients. Let U_1 and U_2 be compact manifolds with $U_1 \subset U_2$. If $i\colon U_1 \to U_2$ is the inclusion, $i_*\colon H_*(U_1) \to H_*(U_2)$ is the group homomorphism induced by i, and u is the fundamental cycle of U_1, then the dual to i_*u in $H^*(U_2)$ will be called the dual to U_1 in U_2, and will be denoted $D(U_2, U_1)$.

Let E be an m-dimensional vector bundle over a compact manifold U. $W(E) = 1 + W_1(E) + \cdots + W_m(E)$ will denote the Whitney class of E. If $\sigma\colon U \to E$ is a transversal section and Z is the zero set of σ, then $W_m(E)$ is the dual to Z in U.

If $A_b(f) = \phi$ for each $b > a$, then $\tilde{s}_a A_a(f)$ is the zero set of $\tilde{\gamma}_a$. Hence the following

Lemma 7.1. *Suppose the fiber dimension of S_a is m. Let $f\colon X \to Y$ be $A_a(\gamma)$-transversal. suppose $A(f)$ is compact and $A_b(f) = \phi$ for each $b > a$. Then the dual to $\tilde{s}_a A_a(f)$ in $G_a(f^{n*}E_2)$ is $W_m(\bar{f}^*S_a)$.*

If dimension $E_2 = 1$, then $G_1(f^{n*}E_2) = A(f)$ and $\bar{f}^*S_1 = (f^{n*}E_2)^* \otimes (f^{n*}E_3)$. Thus

Proposition 7.2. *Let* $\dim E_2 = 1$, $\dim E_3 = m$ *and* $f\colon X \to Y$ *be* $A_1(\gamma)$-*transversal. If $A(f)$ is compact, then*

$$D(A(f), A_1(f)) = W_m((f^{n*}E_2)^* \otimes (f^{n*}E_3)) = \sum_{i=0}^{m} W_1(f^{n*}E_2)^i W_{m-i}(f^{n*}E_3)\ .$$

Let U_1 and U_2 be compact manifolds and let $\phi\colon U_1 \to U_2$ be continuous. ϕ induces a group (not ring) homomorphism $\phi_\#\colon H^*(U_1) \to H^*(U_2)$. $\phi_\#$ is defined by composing ϕ_* with the appropriate duality isomorphisms.

If $\phi^*\colon H^*(U_2) \to H^*(U_1)$ is the ring homomorphism induced by ϕ, $u_1 \in H^*(U_1)$ and $u_2 \in H^*(U_2)$, then $\phi_\#((\phi^*u_2)\cdot u_1) = u_2 \cdot \phi_\# u_1$. If $\phi\colon U_1 \to U_2$ and $\psi\colon U_2 \to U_3$, then $(\psi\phi)_\# = \psi_\#\phi_\#$. Note that if $U_1 \subset U_2$, $i\colon U_1 \to U_2$ is the inclusion, and 1 is the unit cohomology class of U_1, then $D(U_2, U_1) = i_\# 1$.

For the remainder of this section, X will be compact.

Lemma 7.3. *Let E be a vector bundle over X of fiber dimension m. Let $a \le m$ and let $\bar{\pi}\colon G_a(E) \to X$ be the projection. Suppose $0 \to L_a \to \bar{\pi}^*E \to M_a \to 0$ is the usual sequence over $G_a(E)$. Then $\bar{\pi}_\#(W_{m-a}(M_a)^a)$ is the unit cohomology class of X.*

Proof. See [5].

If E is a vector bundle over X, then $-E$ will denote the inverse bundle of E. Porteous uses Lemma 7.3 to prove

Theorem 7.4. *Let X be a compact p-manifold, Y a q-manifold and a a positive integer such that $a \leq p$ and $a > p - q$. Let $f\colon X \to Y$ be $Z(a)$-transversal and suppose $Z_b(f) = \phi$ for $b > a$. Then $D(X, Z_a(f))$ is the determinant of the $a \times a$ matrix whose i, j term is $W_{q-p+a+i-j}(f^*TY - TX)$.*

Proof. See [5].

(Actually, Porteous proves a somewhat stronger theorem.)

Lemma 7.5. *Let E be a vector bundle over X of fiber dimension m, and $\bar{\pi}\colon G_1(E) \to X$ be the bundle projection. Then $\bar{\pi}_\sharp(W_1(L_1)^j) = W_{j-m+1}(-E)$ for each j.*

Proof (By induction on j). Let $a = W_1(L_1)$, $1 + b_1 + \cdots + b_{m-1} = W(M_1)$ and $1 + c_1 + \cdots + c_m = \bar{\pi}^*W(E)$. $\bar{\pi}_\sharp$ lowers dimension by the fiber dimension of $G_1(E)$, so the lemma is trivial for $j < m - 1$. By the Whitney duality theorem, $\sum_{i=0}^{m-1} a^i c_{m-1-i} = b_{m-1}$, so $\bar{\pi}_\sharp b_{m-1} = \sum_{i=0}^{m-1} \bar{\pi}_\sharp(a^i) W_{m-1-i}(E) = \bar{\pi}_\sharp(a^{m-1})$. But by Lemma 7.3, $\bar{\pi}_\sharp b_{m-1} = 1$, so the lemma is valid for $j = m - 1$.

We now assume that $t \geq m - 1$ and that Lemma 7.5 is valid for $j \leq t$, and prove for $j = t + 1$. $\sum_{i=0}^{m-1} a^i c_{m-1-i} = b_{m-1}$ implying $\sum_{i=0}^{m-1} a^{i+1} c_{m-i} = ab_{m-1} = c_m$, so $\sum_{i=1}^{m} a_i c_{m-i} = 0$. Thus if $t + 1 \geq m$, then $\sum_{i=0}^{m} a^{t+1-m+i} c_{m-i} = 0$. Applying $\bar{\pi}_\sharp$ and the induction hypothesis,

$$0 = \bar{\pi}_\sharp(a^{t+1}) + \sum_{i=0}^{m-1} \bar{\pi}_\sharp(a^{t+1-m+i}) W_{m-i}(E)$$

$$= \bar{\pi}_\sharp(a^{t+1}) + \sum_{i=0}^{m-1} W_{t+2-2m+i}(-E) W_{m-i}(E) \; ,$$

so $\bar{\pi}_\sharp(a^{t+1}) = \sum_{i=0}^{m-1} W_{t+2-2m+i}(-E) W_{m-i}(E)$. But $\sum_{i=0}^{m} W_{t+2-2m+i}(-E) W_{m-i}(E)$ is the $(t + 2 - m)$-dimensional term of $W(-E)W(E)$ which is 0 since $(t + 2 - m) \neq 0$. It follows that

$$\bar{\pi}_\sharp(a^{t+1}) = \sum_{i=0}^{m} W_{t+2-2m+i}(-E) W_{m-i}(E) = W_{t+2-m}(-E) \; .$$

Theorem 7.6. *Let $p \leq q$ and $I^n = (\underbrace{1, \cdots, 1}_{n})$. Let X be a compact p-manifold, and Y a q-manifold. Suppose $f\colon X \to Y$ is $Z(I^m)$-transversal for each $m \leq n$ and such that $Z_i(f) = \phi$ for each $i > 1$. Then the dual to $Z(I^n)(f)$ in X is a polynomial in the $W_i(f^*TY - TX)$, and this polynomial is computable and does not depend on X, Y and f.*

Proof. Let all notation be as in § 6. If $1 \leq m \leq n$, then $f^{m*}E_2^m = f^{1*}E_2^1$ and $f^{m*}E_3^m = \left(\bigotimes_m f^{1*}E_2^1\right) \otimes f^{1*}Q^1$ over $Z(I^m)(f)$. Note that $\dim E_2^1 = 1$ and $\dim Q^1 = q - p + 1$. Let $i_m\colon Z(I^m)(f) \to Z(I^1)(f)$ be the inclusion. By Proposition 7.2, if $1 \leq m \leq n - 1$, then

$$D(Z(I^m)(f), Z(I^{m+1})(f)) = i_m^*\left(W_{q-p+1}\left(\left(\bigotimes_m f^{1*}E_2^1\right) \otimes f^{1*}Q^1\right)\right)$$
$$= i_m^*\left(\sum_{i=0}^{q-p+1}\left((m+1)W_1(f^{1*}E_2^1)\right)^i W_{q-p+1-i}(f^{1*}Q^1)\right) .$$

Thus

$$D(Z(I^1)(f), Z(I^n)(f)) = \prod_{m=1}^{n-1}\left\{\sum_{i=0}^{q-p+1}((m+1)W_1(f^{1*}E_2^1))^i W_{q-p+1-i}(f^{1*}Q^1)\right\} .$$

Denote this cohomology class by C. If $i\colon Z(I^1)(f) \to X$ is the injection, then $D(X, Z(I^n)(f)) = i_\# C$. Let $\bar{\pi}\colon G_1(TX) \to X$ be the projection and $s^1\colon Z(I^1)(f) \to G_1(TX)$ the obvious section. Then $\bar{\pi}s^1 = i$ so $D(X, Z(I^n)(f)) = \bar{\pi}_\# s^1{}_\# C$. If $\bar{Q}$ is defined over $G_1(TX)$ by the exactness $0 \to M_1 \to \bar{\pi}^* f^* TY \to \bar{Q} \to 0$, then $f^{1*}Q^1 = s^{1*}\bar{Q}$. As noted before, $f^{1*}E_2^1 = s^{1*}L_1$. $s_\#^1 C$ is now computable by Lemma 7.1. By the Whitney duality theorem, $s_\#^1 C$ is expressible in terms of $W_1(L_1)$ and the Whitney closses of $\bar{\pi}^* TX$ and $\bar{\pi}^* f^* TY$. By Lemma 7.5, $\bar{\pi}_\# s_\#^1 C$ is computable.

Theorem 7.7. *Let $p \geq q$, and $I^n = (p-q+1, \underbrace{1, \cdots, 1}_{n-1})$. Let X be a compact p-manifold, and Y a q-manifold. Suppose $f\colon X \to Y$ is $Z(I^m)$-transversal for each $m \leq n$ and such that*

i) *$Z_i(f) = \phi$ for each $i > p-q+1$, and*

ii) *$Z(p-q+1, i)(f) = \phi$ for each $i > 1$.*

Then the dual to $Z(I^n)(f)$ in X is a polynomial in the $W_i(f^*TY - TX)$, and this polynomial is computable and does not depend on X, Y and f.

Proof. In the spirit of Theorem 7.6.

The author has been unable to find a nice form (as in Theorem 7.4) for the polynomials of Theorems 7.6 and 7.7.

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